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Lattice

We will prove that $(Eq(A) ; \subseteq)$ is a lattice
with $Eq(A)$ being the set of all equivalence relations on A
and A being an arbitrary set.
We will prove this by showing that P and Q hold for any two elements X and Y in $Eq(A)$.
$P \overset{\text{def.}}{\Leftrightarrow}$ the join of X and Y is $(X \circ Y)^*$
$Q \overset{\text{def.}}{\Leftrightarrow}$ the meet of X and Y is $X \cap Y$

proof of P

proof of P:
(i) $\overset{\text{def.}}{\Leftrightarrow}$ for any two equivalence relations σ and π on A, $(\sigma \circ \pi)^*$ is also an equivalence relation on A
proof:
I will prove (i) by checking all conditions of equivalence relations:
Let A be the universe. | by definition of $\cup$

1) $(\sigma \circ \pi)^*$ is reflexive:
$\forall a\, ((a,a) \in \sigma \wedge (a,a) \in \pi)$ (1) (true by reflexivity of σ, π) | $\sigma \subseteq \sigma \circ \pi$ and $\pi \subseteq \sigma \circ \pi$ $\rightarrow$ proof (1)
$\Rightarrow \forall a\, ((a,a) \in \sigma \circ \pi \wedge (a,a) \in \sigma \circ \pi)$
$\Rightarrow \forall a\, ((a,a) \in \sigma \circ \pi)$ | $(\sigma \circ \pi) \subseteq (\sigma \circ \pi)^*$
$\Leftrightarrow \forall a\, ((a,a) \in (\sigma \circ \pi)^*)$
$\Leftrightarrow (\sigma \circ \pi)^*$ is reflexive (2)
As (1) is true, (2) holds.

2) $(\sigma \circ \pi)^*$ is transitive:
let a and b be arbitrary elements in A.
(1) $(\exists c\, ((a,c) \in (\sigma \circ \pi)^* \wedge (c,b) \in (\sigma \circ \pi)^*)$ | existent. instant.
$\Rightarrow (a,c) \in (\sigma \circ \pi)^* \wedge (c,b) \in (\sigma \circ \pi)^*$ | by definition of transitive closure
$\Rightarrow \exists s \exists t\, (s \in \mathbb{N}^+ \wedge t \in \mathbb{N}^+ \wedge (a,c) \in (\sigma \circ \pi)^s \wedge (c,b) \in (\sigma \circ \pi)^t)$ | existential instantiation
$\Rightarrow (a,c) \in (\sigma \circ \pi)^s \wedge (c,b) \in (\sigma \circ \pi)^t$
$\Rightarrow (a,b) \in (\sigma \circ \pi)^s \circ (\sigma \circ \pi)^k$
$\Rightarrow (a,b) \in (\sigma \circ \pi)^{s+k}$ | by definition of trans. closure
$\Rightarrow (a,b) \in (\sigma \circ \pi)^*$ (2)
As (1) $\Rightarrow$ (2) holds, $(\sigma \circ \pi)^*$ is transitive

3) $(\sigma \circ \pi)^*$ is symmetric: let a and b be arbitrary elements in A.
(1) $(a,b) \in (\sigma \circ \pi)^*$
$\Rightarrow \exists k\, ((a,b) \in (\sigma \circ \pi)^k)$ | symmetry of σ and π, proof (2)
$\Rightarrow (b,a) \in (\sigma \circ \pi)^*$ (2)
As (1) $\Rightarrow$ (2) holds, $(\sigma \circ \pi)^*$ is symmetric.

Consequently, (i) holds.
As all X and Y in $Eq(A)$ are equivalence relations, it also holds that $(X \circ Y)^*$ is an equivalence relation on A by (i).
Thus, $(X \circ Y)^* \in Eq(A)$ (ii)
(iii) $\overset{\text{def.}}{\Leftrightarrow}$ $\forall Z ((Z \in Eq(A) \wedge X \subseteq Z \wedge Y \subseteq Z) \rightarrow (X \circ Y)^* \subseteq Z)$
This holds by proof 3.
(iv) $\overset{\text{def.}}{\Leftrightarrow}$ $X \subseteq (X \circ Y)^* \wedge Y \subseteq (X \circ Y)^*$
(iv) holds, as $X \subseteq X \circ Y$ by proof 1 and $X \circ Y \subseteq (X \circ Y)^*$ by definition.
By transitivity of $\subseteq$, $X \subseteq (X \circ Y)^*$.
The same argumentation can be used to show $Y \subseteq (X \circ Y)^*$.
By (ii), (iii) and (iv): P holds.

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proof (1):

$\sigma \subseteq \sigma \circ \pi$

$\Leftrightarrow \quad \forall x \forall y ((x,y) \in \sigma \rightarrow (x,y) \in \sigma \circ \pi))$ | universal instat.

$\Leftrightarrow \quad ((x,y) \in \sigma \rightarrow (\exists z ((x,z) \in \sigma \wedge (z,y) \in \pi)))$

$\Leftarrow \quad ((x,y) \in \sigma \rightarrow ((x,y) \in \sigma \wedge (y,y) \in \pi))$ | reflexivity of π

$\Leftrightarrow \quad ((x,y) \in \sigma \rightarrow (x,y) \in \sigma)$

$\Leftrightarrow \quad ((x,y) \notin \sigma \vee (x,y) \in \sigma)$

The last statement is trivially true.

Thus, $\sigma \subseteq \sigma \circ \pi$.

The same explanation works for the proof of $\pi \subseteq \sigma \circ \pi$.

proof (2):

to be proven: $\forall k \in \mathbb{N}^+ \; (P(k))$

$P(n) \overset{\text{def.}}{\Leftrightarrow} (a,b) \in (\sigma \circ \pi)^k \Rightarrow (b,a) \in (\sigma \circ \pi)^*$ for all a, b in A

with σ, π being equivalence relations on some set A

Let A be the universe.

proof by induction:

base case: $P(1)$ (Let a, b be arbitrary elements in A.)

(i) $(a,b) \in (\sigma \circ \pi)^1$

$\Rightarrow \exists x \, ((a,x) \in \sigma \wedge (x,b) \in \pi)$ | reflexivity of σ and π

$\Rightarrow \exists x \, ((b,b) \in \sigma \wedge (x,b) \in \pi \wedge (a,x) \in \sigma \wedge (a,a) \in \pi)$ | symmetry of σ and π

$\Rightarrow \exists x \, ((b,b) \in \sigma \wedge (b,x) \in \pi \wedge (x,a) \in \sigma \wedge (a,a) \in \pi)$

$\Rightarrow \exists x \, ((b,x) \in \sigma \circ \pi \wedge (x,a) \in \sigma \circ \pi)$

$\Rightarrow (b,a) \in (\sigma \circ \pi)^2$

$\Rightarrow (b,a) \in (\sigma \circ \pi)^*$ (ii)

As (i) $\Rightarrow$ (ii) holds, $P(1)$ holds.

induction hypothesis: $P(n)$ holds for some $n \in \mathbb{N}^+$

induction step:

(i) $(a,b) \in (\sigma \circ \pi)^{n+1}$

$\Rightarrow \exists x \, ((a,x) \in (\sigma \circ \pi)^n \wedge (x,b) \in (\sigma \circ \pi))$ | exist. initialization, IH

$\Rightarrow (x,a) \in (\sigma \circ \pi)^* \wedge (x,b) \in (\sigma \circ \pi)$

$\Rightarrow \exists k \, ((x,a) \in (\sigma \circ \pi)^k \wedge (x,b) \in (\sigma \circ \pi))$ | exist. intialization, base case

$\Rightarrow \exists k \, ((x,a) \in (\sigma \circ \pi)^k \wedge (b,x) \in (\sigma \circ \pi)^2)$

$\Rightarrow \exists k \exists e \, ((x,a) \in (\sigma \circ \pi)^k \wedge (b,e) \in (\sigma \circ \pi) \wedge (e,x) \in (\sigma \circ \pi))$

$\Rightarrow \exists k \exists e \, ((e,a) \in (\sigma \circ \pi)^{k+1} \wedge (b,e) \in (\sigma \circ \pi))$

$\Rightarrow \exists k \, ((b,a) \in (\sigma \circ \pi)^{k+2})$

$\Rightarrow (b,a) \in (\sigma \circ \pi)^*$

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proof 3:

proof by induction: $\forall n \in N^+ \, (P(n))$

with $P(n) \overset{\text{def.}}{\iff} ((X \subseteq Z \land Y \subseteq Z) \Rightarrow (X \circ Y)^n \subseteq Z)$ for all elements X, Y, Z in $Eq(A)$

base case: $P(1)$

proof by contradiction: Suppose that $P(1)$ doesn't hold.

$$
\begin{aligned}
\neg(P(1)) &\iff \exists X \exists Y \exists Z \, ((X, Y, Z) \in Eq(A)^3 \land X \subseteq Z \land Y \subseteq Z \land \neg(X \circ Y) \subseteq Z) && \mid \text{exist. inst.} \\
&\Rightarrow X \subseteq Z \land Y \subseteq Z \land \neg(\forall x \forall y \, ((x, y) \in (X \circ Y) \rightarrow (x, y) \in Z) \\
&\iff X \subseteq Z \land Y \subseteq Z \land \exists x \exists y \, ((x, y) \in (X \circ Y) \land (x, y) \notin Z) && \mid \text{exist. inst.} \\
&\Rightarrow X \subseteq Z \land Y \subseteq Z \land (x, y) \in (X \circ Y) \land (x, y) \notin Z) \\
&\Rightarrow \exists z \, ((x, z) \in X \land (z, y) \in Y) \land (x, y) \notin Z && \mid (x, z) \in Z \text{ by } (x, z) \in X \text{ and } X \subseteq Z \\
& && (z, y) \in Z \text{ by } (z, y) \in Y \text{ and } Y \subseteq Z \\
&\Rightarrow \exists z \, ((x, z) \in Z \land (z, y) \in Z) \land (x, y) \notin Z && \mid \text{transitivity of } Z \ (Z \in Eq(A)) \\
&\Rightarrow (x, y) \in Z \land (x, y) \notin Z
\end{aligned}
$$

The last statement is false. Thus, the assumption must have been wrong.

Therefore, $P(1)$ holds.

induction hypothesis: $P(n)$ holds for some $n \in N^+$

induction step:

proof by contradiction: Suppose that $P(n+1)$ doesn't hold

$$
\begin{aligned}
\neg(P(n+1)) &\iff \exists X \exists Y \exists Z \, ((X, Y, Z) \in Eq(A)^3 \land X \subseteq Z \land Y \subseteq Z \land \neg((X \circ Y)^{n+1} \subseteq Z) && \mid \text{exist. inst.} \\
&\Rightarrow X \subseteq Z \land Y \subseteq Z \land \neg(\forall x \forall y \, ((x, y) \in (X \circ Y)^{n+1} \rightarrow (x, y) \in Z) \\
&\iff X \subseteq Z \land Y \subseteq Z \land \exists x \exists y \, ((x, y) \in (X \circ Y)^{n+1} \land (x, y) \notin Z) && \mid \text{exist. inst.} \\
&\Rightarrow X \subseteq Z \land Y \subseteq Z \land (x, y) \in (X \circ Y)^* \land (x, y) \notin Z) \\
&\Rightarrow X \subseteq Z \land Y \subseteq Z \land \exists z \, ((x, z) \in (X \circ Y)^n \land (z, y) \in (X \circ Y)) \land (x, y) \notin Z && \mid \text{IH, base case} \\
&\Rightarrow \exists z \, ((x, z) \in Z \land (z, y) \in Z) \land (x, y) \notin Z && \mid \text{transitivity of } Z \ (Z \in Eq(A)) \\
&\Rightarrow (x, y) \in Z \land (x, y) \notin Z
\end{aligned}
$$

The last statement is false. Thus, the assumption must have been wrong.

Therefore, $P(n+1)$ must hold (given $P(n)$).

Additionally:

$\forall n \in N^+ \, (P(n)) \Rightarrow ((X \subseteq Z \land Y \subseteq Z) \Rightarrow (X \circ Y)^* \subseteq Z)$ for all X, Y, Z in $Eq(A)$.

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proof of Q

proof of Q:

(i) $\overset{\text{def.}}{\Leftrightarrow}$ for any two equivalence relations σ and π on A, $(\sigma \cap \pi)$ is also an equivalence relation on A

proof:

I will prove this statement by checking all conditions of equivalence relations:

Let σ and π be equivalence relation on A. Let A be the universe.

1) $\sigma \cap \pi$ is reflexive:

(1) $\forall a\,((a,a) \in \sigma \wedge (a,a) \in \pi)$ (by reflexivity of σ, π) | by definition of $\cap$

$\Leftrightarrow \forall a\,((a,a) \in \sigma \cap \pi)$

$\Leftrightarrow \sigma \cap \pi$ is reflexive

As (1) is true by reflexivity of σ and π. Thus, $\sigma \cap \pi$ is reflexive.

2) $\sigma \cap \pi$ is transitive: let a and b be arbitrary elements in A.

(1) $(\exists c\,((a,c) \in \sigma \cap \pi \wedge (c,b) \in \sigma \cap \pi))$ | existential instantiation

$\Rightarrow ((a,c) \in \sigma \cap \pi \wedge (c,b) \in \sigma \cap \pi)$ | by definition of $\cap$

$\Rightarrow (a,c) \in \sigma \wedge (a,c) \in \pi \wedge (c,b) \in \sigma \wedge (c,b) \in \pi$ | by transitivity of σ and π

$\Rightarrow (a,b) \in \sigma \wedge (a,b) \in \pi$ | definition of $\cap$

$\Rightarrow (a,b) \in \sigma \cap \pi\,)$ (2)

As (1) $\Rightarrow$ (2) holds, $\sigma \cap \pi$ is transitive.

3) $\sigma \cap \pi$ is symmetric: let a and b be arbitrary elements in A.

(1) $((a,b) \in \sigma \cap \pi)$ | definition of $\cap$

$\Rightarrow (a,b) \in \sigma \wedge (a,b) \in \pi$ | symmetry of σ and π

$\Rightarrow (b,a) \in \sigma \wedge (b,a) \in \pi$ | definition of $\cap$

$\Rightarrow ((b,a) \in \sigma \cap \pi)$ (2)

As (1) $\Rightarrow$ (2) holds, $\sigma \cap \pi$ is symmetric.

Since $\sigma \cap \pi$ is reflexive, symmetric and transitive, $\sigma \cap \pi$ is an equivalence relation.

Consequently, (i) holds.

As all X and Y in $Eq(A)$ are equivalence relations, it also holds that $X \cap Y$ is an equivalence relation on A by (i).

Thus, $X \cap Y \in Eq(A)$ (ii)

(iii) $\overset{\text{def.}}{\Leftrightarrow}$ $\forall Z((Z \in Eq(A) \wedge Z \subseteq X \wedge Z \subseteq Y) \rightarrow Z \subseteq X \cap Y)$

proof by contradiction:

Suppose that (iii) was false.

$\neg$(iii) $\Leftrightarrow \exists Z\,(Z \in Eq(A) \wedge Z \subseteq X \wedge X \subseteq Y \wedge Z \nsubseteq X \cap Y)$ | exist. instant.

$\Rightarrow Z \subseteq X \wedge Z \subseteq Y \wedge Z \nsubseteq X \cap Y$

$\Leftrightarrow \forall a\,(a \in Z \rightarrow (a \in X \wedge a \in Y)) \wedge Z \nsubseteq X \cap Y$ | definition of $\cap$

$\Rightarrow \forall a\,(a \in Z \rightarrow a \in X \cap Y) \wedge Z \nsubseteq X \cap Y$

$\Rightarrow Z \subseteq X \cap Y \wedge Z \nsubseteq X \cap Y$

The last statement is false. Thus, the assumption must have been false.

Therefore, (iii) must hold.

(iv) $\overset{\text{def.}}{\Leftrightarrow}$ $X \cap Y \subseteq X \wedge X \cap Y \subseteq Y$

proof of (iv):

$X \cap Y \subseteq X \Leftrightarrow \forall x\,(x \in (X \cap Y) \rightarrow x \in X)$

$\Leftrightarrow \forall x\,((x \in X \wedge x \in Y) \rightarrow x \in X)$

$\Leftrightarrow \forall x\,(\neg(x \in X \wedge x \in Y) \vee x \in X)$

$\Leftrightarrow \forall x\,(x \notin X \vee x \in Y \vee x \in X)$

$\Leftrightarrow \forall x\,(x \in Y \vee T)$

$\Leftrightarrow \forall x\,(T)$

The last statement is trivially true.

Thus, $X \cap Y \subseteq X$ holds.

The same argumentation also works for $X \cap Y \subseteq Y$.

Thus, (iv) holds.

By (ii), (iii) and (iv): Q holds.