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# Bi(t)jection

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Let $S = \{0,1\}^{\infty}$ be the set of infinite bit sequences $(b_1, b_2, b_3, \ldots)$.
We will construct injections
$f: [0,1] \to S$ and $g: S \to [0,1]$
By the Schröder-Bernstein theorem, the existence of these two injections implies there is a bijection $[0,1] \to S$

We need to consider that dyadic rationals, rational numbers $p/q$ where $q$ is a power of two, have two binary expansions: one terminating (ending with infinitely many 0s) and one non-terminating (ending with infinitely many 1s).

To show the injection $f: [0,1] \to S$ we will write each $x \in [0,1]$ in binary in the following way:
If $x$ has two binary expansions, choose the one that ends with infinitely many $0$s (the terminating one).
Now record the bits of that expansion as an infinite sequence in $S$ as follows:

- If $x < 1$ then its binary expansion is
  $x = 0.b_1 b_2 b_3 \ldots_2$

Set $f(x) = (0, b_1, b_2, b_3, \ldots)$

- If $x = 1$ then its binary expansion is
$x = 1.000\ldots_2$
Set $f(x) = (1, 0, 0, 0, \ldots)$

This way, each $x$ has exactly one sequence assigned to it.
If $f(x) = f(y)$ then the chosen binary expansions agree bitwise, hence $x = y$.
Thus $f$ is injective.

To show that $g$ is injective you cannot just send each sequence $(b_1, b_2, b_3, \ldots)$ to the binary expansion $0.b_1 b_2 b_3\ldots$ because different sequences can give the same real number (e.g. $0.1000\ldots_2 = 0.01111\ldots_2$).
Define $g$ by embedding the bits of a sequence into the even binary places:

$$g((b_1, b_2, b_3, \ldots)) = \sum_{n=1}^{\infty} b_n \cdot 2^{-2n}$$

Equivalently, the binary expansion of $g((b_n))$ is
$$0.0b_1 0b_2 0b_3\ldots_2$$
so that all odd-positioned fractional bits are $0$ and the bit $b_n$ sits in position $2n$.
Suppose two different sequences $s \neq s'$ are given. Let $k$ be the first index where they differ: $b_k \neq b_k'$. In the binary expansions of $g(s)$ and $g(s')$ the bit at position $2k$

is $b_k$ for $g(S)$ and $b'_k$ for $g(S')$. So those two binary expansions differ at position $2k$, hence the two real numbers are different.

Any binary expansion produced by $g$ has zeros in every odd position, so it cannot equal a different expansion that would require a tail of 1s.

ex: Originally $0.1000\ldots_2$ and $0.0111\ldots_2$ collided because both equal the real number $\frac{1}{2}$.

But with interleaving 0s:

$$g(1,0,0,0,\ldots) = 0.01000\ldots_2 = 2^{-2} = \frac{1}{4}$$

$$g(0,1,1,1,\ldots) = 0.00010101\ldots_2 = 2^{-4} + 2^{-6} + 2^{-8} + \ldots$$

$$= \frac{2^{-4}}{1-2^{-2}}$$

$$= \frac{\frac{1}{16}}{1-\frac{1}{4}}$$

$$= \frac{1/16}{12/16}$$

$$= \frac{1}{12}$$

So now the sequences give two different real numbers in $[0,1]$

Hence $g$ is injective.

We have constructed injections $f:[0,1] \to S$ and $g: S \to [0,1]$. By the Bernstein-Schröder theorem there exists a bijection $[0,1] \to S$.

Therefore $[0,1]$ and $S = \{0,1\}^{\infty}$ are equinumerous.