Challenge 1b

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1) Two be or not to be (alternative solution)

$\exists k\ 2k=(1-0) \Rightarrow \exists k\ k>0 \land k<1\ (k\leq 0 \Rightarrow 2k\leq 0 \text{ and } k\geq 1 \Rightarrow 2k>1)$

Which is clearly false. Thus:

$\nexists k\ 2k=(1-0) \Rightarrow [0]_{\equiv_2} \neq [1]_{\equiv_2}$ (definition)

$\Rightarrow |\{[0]_{\equiv_2},[1]_{\equiv_2}\}| = 2$

$\Rightarrow |\mathbb{N}/\equiv_2| \geq 2\ (\{[0]_{\equiv_2},[1]_{\equiv_2}\} \subseteq \mathbb{N}/\equiv_2)$

$(\mathbb{N};\leq)$ is well-ordered

$\Rightarrow \forall A\ (A\subseteq \mathbb{N} \land A\neq\emptyset) \rightarrow \exists a\ a\in A \land (\forall x\ x\in A \rightarrow a\leq x)$

(def. of well-ordered)

$\Rightarrow \forall A\ A\in \mathbb{N}/\equiv_2 \rightarrow \exists a\ a\geq 0 \land a\in A \land (\forall x\ x\geq 0 \rightarrow (x\in A \rightarrow a\leq x))$ (1)

$(A\in \mathbb{N}/\equiv_2 \Rightarrow A\subseteq\mathbb{N} \land A\neq\emptyset)$

$\forall n\ n\geq 2 \rightarrow (n-2\geq 0 \land n-(n-2)=2\cdot 1)$

$\Rightarrow \forall n\ n\geq 2 \rightarrow (n-2\geq 0 \land n\equiv_2 n-2)$

$\Rightarrow \forall n\ n\geq 2 \rightarrow (n-2\geq 0 \land (n-2)\in[n]_{\equiv_2})$ (2)

$S \overset{\text{def}}{\Leftrightarrow} \exists n\ n\geq 0 \land n\not\equiv_2 0 \land n\not\equiv_2 1$

Assume S true:

$S \Rightarrow \exists n\ n\geq 0 \land n\neq 0 \land n\neq 1 \land n\not\equiv_2 0 \land n\not\equiv_2 1\ (a=b \Rightarrow a\equiv_2 b)$

$\Rightarrow \exists n\ n\geq 2 \land n\not\equiv_2 0 \land n\not\equiv_2 1\ (a\geq b \land a\neq b \Rightarrow a>b \Rightarrow a\geq b+1$ since $b+1$ covers $b)$

$\Rightarrow \exists n\ \forall x (x\geq 0 \rightarrow (x\in[n]_{\equiv_2} \rightarrow x\geq 2))$

$\Rightarrow \exists n\ \exists a (a\geq 0 \land a\in[n]_{\equiv_2} \land a\geq 2 \land \forall x (x\geq 0 \rightarrow (x\in[n]_{\equiv_2} \rightarrow a\leq x)))$ (1)

$\Rightarrow \exists n\ \exists a (a\geq 2 \land a\in[n]_{\equiv_2} \land \forall x (x\geq 0 \rightarrow (x\in[n]_{\equiv_2} \rightarrow a\leq x)) \land a-2\geq 0 \land a-2\in[n]_{\equiv_2})$ (2)

$\Rightarrow \exists n\ \exists a\ a-2\geq 0 \land a-2<a \land a\leq a-2$

$\Rightarrow \exists a\geq 0\ \exists b\geq 0\ a\leq b \land \neg(a\leq b)\ (a<b \Leftrightarrow \neg(b\leq a))$

which is obviously false $(F\land\neg F\equiv\bot)$

By contradiction, S is false.

$\neg S \Leftrightarrow \forall n\geq 0\ (n\equiv_2 0 \lor n\equiv_2 1)$ $(\neg\exists x\ P(x) \Leftrightarrow \forall x\ \neg P(x)$ + De Morgan's rules$)$

$\Rightarrow \forall n\geq 0\ (n\in[0]_{\equiv_2} \lor n\in[1]_{\equiv_2})$

$\Rightarrow |\mathbb{N}/\equiv_2| \leq 2$

$\Rightarrow |\mathbb{N}/\equiv_2| = 2$ $(|\mathbb{N}/\equiv_2|\geq 2$ + antisymmetry$)$

$\square$