Challenge 2

Wednesday, 30 October 2024 19:09

2) Counting the uncountable

$Q = \{(A, f) \in (\mathcal{P}(B) \times B^B) \mid f \in A^A \wedge A \preceq \mathbb{N} \wedge A \not\prec \mathbb{N}\}$

$(A_1, f_1) \sim (A_2, f_2) \overset{\text{def}}{\Leftrightarrow} \exists g : A_1 \rightarrowtail\!\!\!\!\twoheadrightarrow A_2 \text{ s.t. } f_1 = g^{-1} \circ f_2 \circ g$

Reflexive:

$\forall (A, f) \in Q$:

id_A is a bijection from A to A, and

$id_A^{-1} \circ f \circ id_A = id_A \circ f \circ id_A = f$ (Since $id = \hat{id}$ and $p \circ id = p$)

Hence: $(A, f) \sim (A, f)$ so $\sim$ is reflexive

Symmetric:

$\forall (A_1, f_1), (A_2, f_2) \in Q$:

$(A_1, f_1) \sim (A_2, f_2) \Leftrightarrow \exists g : A_1 \to A_2$, g bijective and $f_1 = g^{-1} \circ f_2 \circ g$

$\Leftrightarrow g \circ f_1 = f_2 \circ g$ ($g \circ g^{-1} = g^{-1} \circ g = id$)

$\Leftrightarrow g \circ f_1 \circ g^{-1} = f_2$ (—— '' ——)

$\Leftrightarrow f_2 = (g^{-1})^{-1} \circ f_1 \circ g^{-1}$ ($(g^{-1})^{-1} = g$)

$\Rightarrow \exists g : A_1 \to A_2$, g bijective and $f_2 = g^{-1} \circ f_1 \circ g$

(The inverse of a bijection is also bijective:

f well-defined $\Rightarrow f^{-1}$ injective

f totally-defined $\Rightarrow f^{-1}$ surjective

f injective $\Rightarrow f^{-1}$ well-defined

f surjective $\Rightarrow f^{-1}$ totally-defined)

$\Rightarrow (A_2, f_2) \sim (A_1, f_1)$ (definition)

Transitive:

$\forall (A_1, f_1), (A_2, f_2), (A_3, f_3) \in Q$

$(A_1, f_1) \sim (A_2, f_2) \wedge (A_2, f_2) \sim (A_3, f_3)$

$\Leftrightarrow (\exists g : A_1 \to A_2 \quad f_1 = g^{-1} \circ f_2 \circ g) \wedge (\exists g : A_2 \to A_3 \quad f_2 = g^{-1} \circ f_3 \circ g)$ (definition)

$\Leftrightarrow \exists g : A_1 \to A_2 \, \exists h : A_2 \to A_3 \; f_1 = g^{-1} \circ f_2 \circ g \wedge f_2 = h^{-1} \circ f_3 \circ h$ ($\exists x P(x) \wedge \exists y Q(y) \equiv \exists x \exists y P(x) \wedge Q(y)$)

$\Rightarrow \exists g : A_1 \to A_2 \, \exists h : A_2 \to A_3 \; f_1 = g^{-1} \circ (h^{-1} \circ f_3 \circ h) \circ g$ ($f = g \Leftrightarrow h \circ f = h \circ g$)

$\Rightarrow \exists g : A_1 \to A_2 \, \exists h : A_2 \to A_3 \; f_1 = (h \circ g)^{-1} \circ f_3 \circ (h \circ g)$ (associativity of $\circ$ + $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$)

$\Rightarrow \exists g : A_1 \to A_3 \; f_1 = g^{-1} \circ f_3 \circ g$ (A composition of bijections is also bijective)

$\Rightarrow (A_1, f_1) \sim (A_3, f_3)$ (definition)

[illegible]

$\Rightarrow (A_2, f_1) \sim (A_3, f_3)$ (definition)

$\sim$ is thus an equivalence relation.

$\forall (A, f) \in Q$:

$A \preceq \mathbb{N} \land A \not\sim \mathbb{N} \Leftrightarrow A$ is finite (Theorem 3.17)

$\forall (A_1, f_1), (A_2, f_2) \in Q$:

$(A_1, f_1) \sim (A_2, f_2) \Rightarrow$ There is a bijection $g: A_1 \to A_2$ (definition)

$\Leftrightarrow A_1 \sim A_2$ (def 3.42) $\Leftrightarrow |A_1| = |A_2|$ (A_1, A_2 finite)

Consider, $\forall n \in \mathbb{N}$, a set $S_n \subseteq B$ s.t. $|S_n| = n$ (S_n is arbitrary but uniquely defined)

$\forall A \subseteq B$:

$|A| = |S_n| \Leftrightarrow A \sim S_n$ (finite sets) $\Leftrightarrow \exists t: A \to S_n$ with t bijective (def 3.42)

$\Rightarrow \forall f \in A^A: (A, f) \sim (S_n, t^{-1} \circ f \circ t) \Rightarrow [(A, f)]_\sim \in Q_n/\sim$, where $Q_n = \{S_n\} \times S_n^{S_n}$

Hence: $Q/\sim = \bigcup_{n \in \mathbb{N}} Q_n/\sim$

$Q_n/\sim \preceq Q_n$ since we can easily map each equivalence class to one of its elements through an injective function $f: Q_n/\sim \to Q_n$:

$f(E) = f(E') \Rightarrow \exists a \; a \in E \land a \in E'$

$\Rightarrow \forall x \; x \in E \Leftrightarrow x \sim a \Leftrightarrow x \in E'$

$\Rightarrow E = E'$

$|Q_n| = |\{S_n\}| \cdot |S_n^{S_n}| = |S_n^{S_n}| \Rightarrow Q_n \sim S_n^{S_n}$

$|S_n^{S_n}| = |n|^{|n|}$ (for each of the n elements of S_n, you can choose amongst n images in S_n)

$S_n^{S_n}$ is finite $\Rightarrow S_n^{S_n} \preceq \mathbb{N}$ (Theorem 3.17)

$\Rightarrow Q_n/\sim \preceq \mathbb{N}$ (Transitivity of $\preceq$)

Thus:

$Q/\sim = \bigcup_{i \in \mathbb{N}} Q_n/\sim \preceq \mathbb{N}$ by theorem 3.22 ii, since $Q_1/\sim, Q_2/\sim, \ldots$ form a countable list of countable sets.

By the transitivity of $\preceq$:

$Q/\sim \preceq \mathbb{N}$

$\square$