Challenge 3

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3) Linear Combinations

Let L be the set of all finite rational linear combinations of elements in S:

$$L = \{\sum_{i=1}^{m} s_i \cdot q_i \mid n \in \mathbb{N} \land \forall 1 \leq i \leq n\ (s_i, q_i) \in S \times \mathbb{Q}\}$$

Let $\approx$ be the equivalence defined on $(S \times \mathbb{Q})^*$ s.t.:

$\forall (s_i, q_i)_{i=1}^{m}, (s'_i, q'_i)_{i=1}^{n} \in (S \times \mathbb{Q})^*$:

$$(s_i, q_i)_{i=1}^{m} \approx (s'_i, q'_i)_{i=1}^{n} \overset{\text{def}}{\iff} \sum_{i=1}^{m} s_i \cdot q_i = \sum_{i=1}^{n} s'_i \cdot q'_i$$

Obviously $\approx$ is an equivalence relation, since $=$ is too.

Furthermore, we can construct a bijective function $f : (S \times \mathbb{Q})^* / \approx \rightarrow L$ that maps each equivalence class $E \in (S \times \mathbb{Q})^* / \approx$ to the element $a \in L$ s.t. $\forall (s_i, q_i)_{i=1}^{m} \in E$: $\sum_{i=1}^{m} s_i q_i = a$. f is totally- and well-defined as each equivalence class contains at least one element, and all elements of E are equivalent through $\approx$, and hence give the same linear combination $a \in L$.

f is also injective:

$\exists E, E' \in (S \times \mathbb{Q})^* / \approx : f(E) = f(E')$

$\Rightarrow \exists (s_i, q_i)_{i=1}^{m} \in E,\ (s'_i, q'_i)_{i=1}^{n} \in E' : \sum_{i=1}^{m} s_i \cdot q_i = f(E) = f(E') = \sum_{i=1}^{n} s'_i \cdot q'_i$ (def of f)

$\Rightarrow \exists a \in E, a' \in E' : a \approx a'$ (def of $\approx$)

$\Rightarrow \exists a \exists a' : \forall x\ \ x \in E \Leftrightarrow x \approx a \land a \approx a' \land x \in E' \Leftrightarrow x \approx a'$ (def of equivalence class)

$\Rightarrow \forall x\ x \in E \Leftrightarrow x \in E'$ (transitivity) $\Rightarrow E = E'$

Lastly, f is surjective:

$a \in L \Leftrightarrow \exists (s_i, q_i)_{i=1}^{m} \in (S \times \mathbb{Q})^* : \sum_{i=1}^{m} s_i \cdot q_i = a$ (definition of L)

$\Rightarrow f([(s_i, q_i)_{i=1}^{m}]_{\approx}) = a$

$([(s_i, q_i)_{i=1}^{m}]_{\approx} \in (S \times \mathbb{Q})^* / \approx$ since $(s_i, q_i)_{i=1}^{m} \in (S \times \mathbb{Q})^*)$

$[L(s_i, q_i)_{i=1}^m]_\approx \in (S \times Q)^*/\approx$ since $(s_i, q_i)_{i=1}^m \in (S \times Q)^*$

Thus: $L \sim (S \times Q)^*/\approx$ (def 3.42) $\Rightarrow L \preceq (S \times Q)^*/\approx$

Furthermore, the function $g : (S \times Q)^*/\approx \to (S \times Q)^*$, which maps an arbitrarily fixed element x belonging to E to E, is totally- and well-defined since each $E \in (S \times Q)^*/\approx$ has at least one element and by definition we fix a unique, arbitrary $x \in E$ s.t. $f(E) = x$.

g is injective: $g(E) = g(E') \Rightarrow g(E) \in E \wedge g(E) \in E'$ (def of g)

$\Rightarrow \forall x \; x \in E \Leftrightarrow x \approx g(E) \Leftrightarrow x \in E'$

$\Rightarrow E = E'$

Thus $(S \times Q)^*/\approx \preceq (S \times Q)^* \Rightarrow L \preceq (S \times Q)^*$ (transitivity)

$S \sim \mathbb{N} \wedge Q \sim \mathbb{N} \Rightarrow S \times Q \sim \mathbb{N} \times \mathbb{N}$ (3.21)

$\Rightarrow S \times Q \sim \mathbb{N}$ (3.19)

By theorem 3.22 iii

$(S \times Q)^* \preceq \mathbb{N} \Rightarrow L \preceq \mathbb{N}$ (transitivity)

$\square$