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DiscMath feat. LinAlg

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Let $A = \{a_1, \dots, a_n\}$ be a finite set and $\rho \subseteq A \times A$ a relation.
Fix the ordering $(a_1, \dots, a_n)$ of $A$.
Define the adjacency matrix $M = (m_{ij}) \in \{0,1\}^{n \times n}$ by

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, a_j) \in \rho \\ 0 & \text{otherwise} \end{cases}$$

Recall $\rho^k$ is the k-fold composition of $\rho$ with itself.
$(a_i, a_j) \in \rho^k$ iff there exist $(b_1, \dots, b_{k-1}) \in A$ with
$(a_1, b_1) \in \rho, (b_1, b_2) \in \rho, \dots, (b_{k-1}, a_j) \in \rho$.

To reflect composition we use the boolean matrix product.
For two $\{0,1\}$-matrices $A = (a_{ij})$ and $B = (b_{ij})$ define their boolean product $Z = A \odot B$ by

$$z_{ij} = \bigvee_{r=1}^{n} (a_{ir} \wedge b_{rj})$$

i.e. $z_{ij} = 1$ iff there exists $r$ with $a_{ir} = 1$ and $b_{rj} = 1$.

Claim:
If $M$ is the adjacency matrix of $\rho$, then the k-th boolean power $M^{\odot k}$ (that is multiplying $M$ with itself k-times using $\odot$) is the adjacency matrix of $\rho^k$. Concretely

$$(M^{\odot k})_{ij} \iff (a_i, a_j) \in \rho^k$$

Proof by induction:

Base $k=0$:

$\rho^0 = id_A$. By definition $M^{\odot 0} = I_n$. The entry $(I_n)_{ij} = 1$ iff $i=j$, which matches $(a_i, a_j) \in id_A$. So the statement holds for $k=0$

Base $k=1$:

$M^{\odot 1} = M$. By construction $(M)_{ij} = 1$ iff $(a_i, a_j) \in \rho$. So the statement holds for $k=1$

Induction hypothesis:

Assume the statement holds for an arbitrary $k \in \mathbb{N}$, i.e. $(M^{\odot k})_{ir} = 1$ iff $(a_i, a_r) \in \rho^k$ for all $i, r$.

Inductive step:

Consider $k+1$. By definition of the boolean product:

$$(M^{\odot(k+1)})_{ij} = (M^{\odot k} \odot M)_{ij} = \bigvee_{r=1}^{n} ((M^{\odot k})_{ir} \wedge m_{rj})$$

By the inductive hypothesis $(M^{\odot k})_{ir} = 1$ exactly when $(a_i, a_r) \in \rho^k$, and $m_{rj} = 1$ exactly when $(a_r, a_j) \in \rho$. Therefore the right-hand side is $1$ precisely when there exists $r$ with $(a_i, a_r) \in \rho^k$ and $(a_r, a_j) \in \rho$, so when $(a_i, a_j) \in \rho^{k+1}$.

This proves the claim for $k+1$. By induction the claim holds for all $k \in \mathbb{N}$.

Thus, the identity

$$\left(M^{\odot(k+1)}\right)_{ij} = \left(M^{\odot k} \odot M\right)_{ij}$$

is a recurrence relation. The adjacency matrix for $\rho^{k+1}$ is obtained by taking the adjacency matrix for $\rho^{k}$ and performing one more boolean multiplication by $M$.
To obtain the adjacency matrix $M^{\odot k}$ you simply start at $M$ and apply the step $k-1$ times.