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Repeating Relations

case 1: k = 1

If $k = 1$:

$\rho = \varnothing$

$\rho^1 = \rho = \varnothing$

$\forall i,j \in \{1\} \ (i \neq j \rightarrow \rho^i \neq \rho^j)$

is trivially true, as $i \neq j$ can never be true with $i,j \in \{1\}$, as $\{1\}$ is a singleton.

case 2: k >= 2

If $k \geq 2$:

$A = \{1, 2, 3, \dots, n\}$ with $n \in \mathbb{N}^+$

Let ρ be a relation on A with:

$(a,b) \in \rho \overset{\text{def.}}{\Longleftrightarrow} a + 1 \equiv_{k-1} b$

Then, $\forall i \in \mathbb{N} \ ((a,b) \in \rho^i \Leftrightarrow a + i \equiv_{k-1} b)$. (1)

proof by induction of: $\forall n \in \mathbb{N}^+ (P(n))$ with:

$P(i) \overset{\text{def.}}{\Longleftrightarrow} ((a,b) \in \rho^i \Leftrightarrow a + i \equiv_{k-1} b)$, for all a, b, k in A, for all $n \in \mathbb{N}^+$

base case: $n = 1$

$P(1) \Leftrightarrow ((a,b) \in \rho \Leftrightarrow a + 1 \equiv_{k-1} b)$

This is true by the definition of ρ.

induction hypothesis: $P(n)$ holds for arbitrary $n \in \mathbb{N}^+$

induction step:

$(a,b) \in \rho^{n+1}$

$\Leftrightarrow (a,b) \in \rho^n \circ \rho$

$\Leftrightarrow \exists c \ ((a,c) \in \rho^n \wedge (c,b) \in \rho)$ | exist. instantiation

$\Leftrightarrow (a,c) \in \rho^n \wedge (c,b) \in \rho$ | IH

$\Leftrightarrow a + n \equiv_{k-1} c \wedge c + 1 \equiv_{k-1} b$

$\Leftrightarrow a + n + 1 \equiv_{k-1} c + 1 \wedge c + 1 \equiv_{k-1} b$ | $\equiv_{k-1}$ is an equivalence relation

$\Leftrightarrow a + n + 1 \equiv_{k-1} b$

Thus, $P(n) \Rightarrow P(n+1) \Leftrightarrow ((a+b) \in \rho^{n+1} \Leftrightarrow a + n + 1 \equiv_{k-1} b)$ holds.

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Now, I will show that:

$\rho^a = \rho^b$ (i) $\iff$ $a \equiv_{k-1} b$ (ii) for all $a, b, k \in A$, $n \in \mathbb{N}^+$ (2)

(i) $\Rightarrow$ (ii): by direct proof

Suppose that (i) holds.

(i) $\rho^a = \rho^b$ (The universe is A:) | for any two sets A and B: $A = B \iff ((A \subseteq B) \wedge (B \subseteq A))$

$\Rightarrow$ $\forall x \forall y \, ((x,y) \in \rho^a \leftrightarrow (x,y) \in \rho^b)$ | (1)

$\iff$ $\forall x \forall y \, (x + a \equiv_{k-1} y \leftrightarrow x + b \equiv_{k-1} y)$

$\iff$ $\forall x \forall y \, ((x + a \equiv_{k-1} y \wedge x + b \equiv_{k-1} y) \vee (x + a \not\equiv_{k-1} y \wedge x + b \not\equiv_{k-1} y))$ | $\equiv_{k-1}$ is an equivalence relation, $(k-1) \mid ((x+a) - (x+b))$ $\Rightarrow (k-1) \mid (a-b)$

$\iff$ $\forall x \forall y \, ((a \equiv_{k-1} b) \vee (x + a \not\equiv_{k-1} y \wedge x + b \not\equiv_{k-1} y))$

Now, choose $x = k-1$ and $y = a$, both of which are in A. Then:

(i) $\Rightarrow$ $(a \equiv_{k-1} b) \vee (k-1+a \not\equiv_{k-1} a \wedge x + b \not\equiv_{k-1} y)$ | (3), $(k-1) \mid k-1+a-a$

$\iff$ $(a \equiv_{k-1} b) \vee (\perp \wedge x + b \not\equiv_{k-1} y)$ | domination & identity law

$\Rightarrow$ $(a \equiv_{k-1} b)$

$\iff$ (ii)

Thus, (i) $\Rightarrow$ (ii) holds.

(ii) $\Rightarrow$ (i): by direct proof.

Suppose that (ii) holds. (The universe is A:)

(i) $\iff$ $\forall x \forall y \, ((x,y) \in \rho^a \leftrightarrow (x,y) \in \rho^b)$

$\iff$ $\forall x \forall y \, (x + a \equiv_{k-1} y \leftrightarrow x + b \equiv_{k-1} y)$ | (ii), (3)

$\iff$ $\forall x \forall y \, (x + a \equiv_{k-1} y \leftrightarrow x + a \equiv_{k-1} y)$ | $p \vee \neg p \equiv T$

$\iff$ $\forall x \forall y \, (T)$

The last statement is true.

Thus, (ii) $\Rightarrow$ (i) holds.

As $1 \equiv_{k-1} k$, by (2), $\rho = \rho^k$

Moreover, $\forall i, j \in \{1, \ldots, k-1\} \, (i \neq j \rightarrow \rho^i \neq \rho^j)$

This can be shown by contradiction.

Suppose that this was false. Then, there would be i, j in $\{1, \ldots, k-1\}$ with:

$i \neq j$ and $\rho^i = \rho^j$.

However, $i \neq j$ and $i, j \in \{1, \ldots, k-1\}$

$\Rightarrow$ $i \not\equiv_{k-1} j$ | (2)

$\Rightarrow$ $\rho^i \neq \rho^j$

| explanation by contradiction:
Suppose that $i \equiv_{k-1} j$ and $i, j \in \{1, 2, \ldots, k-1\}$ and $i \neq j$
Then, $(i-j) = a(k-1)$ with some $a \in \mathbb{Z}$.
However, $-k+2 \leq i-j \leq k-2$
Thus, a would have to be 0, meaning that $i - j = 0$
$\Rightarrow i = j$
Therefore, $i \equiv_{k-1} j \wedge i, j \in \{1, \ldots, k-1\} \wedge i \neq j$
is unsatisfiable

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(used step:)

(3) proof: $a \equiv_m b$ and $c \equiv_m d \Rightarrow a + c \equiv_m b + d$

(with a, b, c, d, m being arbitrary elements in $\mathbb{N}$, $m \in \mathbb{N}^+$; The universe is $\mathbb{Z}$.)

$$
\begin{aligned}
a \equiv_m b &\Leftrightarrow m \mid (a - b) \\
&\Leftrightarrow \exists k\,(a - b = mk) && \mid \text{exist. instantiation} \\
&\Leftrightarrow a - b = mk \quad \text{(i)} \\
c \equiv_m d &\Leftrightarrow m \mid (c - d) \\
&\Leftrightarrow \exists e\,(c - d = mk) && \mid \text{exist. instantiation} \\
&\Leftrightarrow c - d = me \quad \text{(ii)} \\
(a + c) - (b + d) &= (a - b) + (c - d) && \mid \text{(i), (ii)} \\
&= mk + me \\
&= m \cdot (k + e) \\
&\Rightarrow m \mid (a + c) - (b + d) \\
&\Rightarrow a + c \equiv_m b + d
\end{aligned}
$$