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Taming the tuple

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$$A^* = \bigcup_{n=1}^{\infty} A^n = A \cup A^2 \cup A^3 \cup A^4 \cup \dots$$

The tuple (a,b) can be <u>uniquely defined</u> as the set $\{\{a\},\{a,b\}\}$, where (a,b) is an element of $A \times A = A^2$

Proof:

Assume: $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$

Equality of the two 2-element sets means the two members on the left are the two members on the right (possibly in swapped order). Hence there are exactly two possibilities:

(i) $\{a\} = \{c\}$ and $\{a,b\} = \{c,d\}$

From $\{a\} = \{c\}$ we get $a = c$. Substituting into $\{a,b\} = \{c,d\}$ yields $\{a,b\} = \{a,d\}$. If $b \neq d$ then $\{a,b\}$ and $\{a,d\}$ are different sets, a contradiction. Therefore $b = d$, thus $a = c$ and $b = d$.

(ii) $\{a\} = \{c,d\}$ and $\{a,b\} = \{c\}$

From $\{a\} = \{c,d\}$ we see the two-element set $\{c,d\}$ is actually a singleton, so $c = d$. Therefore $\{c,d\} = \{c\} = \{a\}$, and so $a = c = d$.
Now $\{a,b\} = \{c\} = \{a\}$ implies $\{a,b\} = \{a\}$, hence $b = a$.
Combining these equalities gives $a = c$ and $b = d$ (in fact all four are equal in this case)

Define $\text{pair}(x,y) := \{\{x\},\{x,y\}\}$

Define a map enc that sends each finite (nonempty) sequence in A to a set built only from elements of A, $\emptyset$, and repeated applications of pair, by recursion on the length of the sequence.

For any $a \in A$ define the encoding of the length-1 sequence (a) by

$$enc(a) = pair(a, \emptyset)$$

For $n \geq 2$ and a sequence $(a_1, a_2, \dots, a_n) \in A^n$ define

$$enc((a_1, \dots, a_n)) = pair(a_1, enc((a_2, \dots, a_n)))$$

Thus the encoding is uniform. The head of the sequence is stored as the first component of a pair and the tail is stored (recursively) as the second component. By construction the second component of a length-1 sequence is $\emptyset$. Every sequence of length > 1 has a non-empty second component.

Define $S = \{enc(s) : s \in A^n, n \geq 1\}$

This S is the set of all nonempty finite sequences from A.

We want to prove that every $enc(s) \in S$ is a set built only from elements of A, $\emptyset$, and repeated pair applications.

Proof by induction:

Base: $n = 1$

For $s = (a)$ we have $enc((a)) = pair(a, \emptyset) = \{\{a\}, \{a, \emptyset\}\}$.

this set is built from the element $a \in A$ and $\emptyset$, so it satisfies the required form.

Inductive hypothesis:
Assume for some $k \geq 1$ that every sequence $t \in A^k$, $enc(t)$ is built only from elements of A, $\emptyset$, and pair applied to such objects.

Inductive step:
Let $s = (a_1, \dots, a_{k+1}) \in A^{k+1}$. By definition
$$enc(s) = pair(a_1, enc((a_2, \dots, a_{k+1}))).$$
By the induction hypothesis $enc((a_2, \dots, a_{k+1}))$ has the required form, and $a_1 \in A$. Thus $pair(a_1, enc(tail))$ is again a set formed by pair applied to acceptable objects, so it also has the required form.

This completes the inductive proof.

We also need to show that the encoding is injective.
That is, if $enc(s) = enc(t)$ then $s = t$. In particular sequences of different lengths have different encodings.
Proof by induction:
Let s and t be two nonempty finite sequences from A with $enc(s) = enc(t)$.
Write $s = (a_1, \dots, a_n)$ and $t = (b_1, \dots, b_m)$ with $n, m \geq 1$.

Base: $n = 1$
Then $enc(s) = pair(a_1, \emptyset)$. Since $enc(t) = enc(s)$, we have

$\mathrm{pair}(b_1, \dots) = \mathrm{pair}(a_1, \emptyset)$. By the unique definition shown at the beginning $b_1 = a_1$ and second component of $\mathrm{enc}(t) = \emptyset$.

But by the recursive definition, the second component of $\mathrm{enc}(t) = \emptyset$ if and only if t has length 1. Hence $m = 1$ as well and $t = (b_1) = (a_1) = s$

Inductive step:

Suppose the claim holds for all sequences of length n. Let s have length $n+1$. Then

$$\mathrm{enc}(s) = \mathrm{pair}(a_1, \mathrm{enc}(\mathrm{tail}_s))$$

and similarly $\mathrm{enc}(t) = \mathrm{pair}(b_1, \mathrm{enc}(\mathrm{tail}_t))$.

Equality of the encodings and the uniqueness property give

$$a_1 = b_1 \text{ and } \mathrm{enc}(\mathrm{tail}_s) = \mathrm{enc}(\mathrm{tail}_t)$$

Now tail_s has length n. By the induction hypothesis applied to the tails, $\mathrm{tail}_s = \mathrm{tail}_t$. Together with $a_1 = b_1$ we get $s = t$.

This completes the inductive proof.

Thus enc is injective.

Conclusion:

The recursive map enc gives a unique set-theoretic representative for every nonempty finite sequence from A.